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THE STABILITY OF VISCOELASTIC PERFECT COLUMNS: A DYNAMIC APPROACH

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Abstract—In this article, the dynamic approach for stability analysis is used to obtain an approximate closed-form expression for the "viscoelastic critical load" of perfect columns made of a linear three-element model material. It is shown that a "critical time" for such structures cannot be defined independently of the type and nature of the disturbance causing instability in the sense of Lyapunov. Criteria are introduced to define a critical time that is independent of the magnitude of the disturbance. The nature of the disturbance, whether it is dynamic or static, however, still has a significant influence on the value of such a critical time.

1. INTRODUCTION

Although most commonly used engineering solid materials are inherently time dependent in their behaviour, various "time-independent" material models have served as well through the years and have helped in predicting the short-term behaviour of structures subjected to various loads and environmental effects. Only under special in-service conditions or for structures made of particular materials, as, for example, metal structures under high-temperature environments or long-span concrete structures undergoing creep and shrinkage, was the design modified to take into account, implicitly or explicitly, such time dependency.

In more recent years, time-dependent effects of material behaviour have become more and more important for a number of reasons, including improved construction techniques, increasing number of high-temperature environments, the appearance on the market of various plastic and fibre-reinforced structural materials, the availability of large-scale computers facilitating numerical solutions and the development of constitutive theory in general continuum mechanics, to mention but a few. One area of research indicating a high level of activity during the last few years deals with the stability of structures made of various time-dependent materials, including linearly viscoelastic structures. Because of the time effects and the peculiar nature in which time enters as a "destabilising" parameter, viscoelastic and creep stability problems are still very much a topic for current research. Investigators are grappling with the application of basic concepts and notions of stability to structures made of such materials and with methods of solution for such problems once they have been defined by an appropriate equation or set of equations. Choice of an appropriate constitutive law describing the time-dependent behaviour of the material in a realistic manner is another major problem facing researchers in this area. Many times, experience with stability behaviour of structures made of time-independent materials leads one to apply methods and techniques that either are inconvenient and cumbersome or, in the worst case, lead to erroneous results and conclusions.

The most natural, fundamental and at the same time most general definition of stability involves notions of motion and time, and is referred to as the "dynamic stability criterion," introduced by Lyapunov. Owing to its generality, this method of analysis is applicable to the widest class of stability problems. For the same reason, however, it also represents a fairly sophisticated, complex mathematical analysis problem[1]. Because of its relative complexity, engineers and researchers have resorted to other methods, less general but simpler in their applications[2, 3]. Methods and criteria, such as the neutral equilibrium of adjacent deflected configurations[4, 5], the second variation of the total potential energy[6, 7] and the initial imperfection approach[8, 9], are commonly used approaches in stability analyses. Undoubtedly, any one of these methods of stability analysis must be considered approximate as compared with the dynamic method as a consequence of the inherent "continuous" motion present in any loaded viscoelastic structure. Results and conclusions obtained from such analyses must be evaluated carefully and cautiously and, if possible, should be compared with corresponding results obtained from a dynamic approach. For example, in [10] a perfect viscoelastic column was subjected to a constant compressive load, and it was concluded that such a column would buckle after some time, referred to as the "critical time." The concept of critical time was arrived at on the basis of assuming that the bending stiffness, EI, of the column is decreasing in time owing to a deterioration of the material constant, E, as a result of creep in the axial direction. The analysis failed to recognize that deterioration in axial stiffness does not necessarily imply a decrease in flexural stiffness, keeping in mind that common viscoelastic material models, describing the behaviour of actual time-dependent materials, invariably respond to any change in loading or loading rate with a response typical of the initial response of the material. Thus, a lateral deflection could not be initiated on such a column without precipitating a response similar to the response of the structure at time $t = 0$. The existence or nonexistence of critical times for perfect structures subjected to axial compressive forces has been the subject of numerous publications and remains to be a topic under investigation[10–13].

The purpose of this study is to review specific features of the stability behaviour of perfect columns made of linear viscoelastic materials, thereby, it is hoped, contributing to a better understanding of this phenomenon. To keep the treatment relatively simple and tractable, a three-element material model is used and applied to the analysis of a simply supported perfect column subjected to a constant concentric axial load. By applying a rigorous Lyapunov-type dynamic analysis, it is shown that the stability of such structures depends only on the material model. A closed-form expression in terms of a rapidly convergent power series is derived for the viscoelastic critical load, P_v, also referred to as the "safe load limit"[14]. The results obtained show that P_v, which is lower than the corresponding Euler load, P_E, is nonzero for structures made of "solid-type materials"[15] with bounded viscosity, whereas $P_v = 0$ for columns made of materials with unlimited viscous deformation, referred to as "fluid-type" materials. However, as was pointed out in [15], even for columns made of such fluid-type materials, there may be a useful service life for the structure during which it can safely carry a given load. The determination of such a "safe service period," which arbitrarily may be referred to as "critical time," is of particular interest and significance to the designer.

For columns made of solid-type materials, equilibrium is unstable for $P > P_v$. Nevertheless, there may again be a period of time during which the column can safely carry such loads without excessive deformations. This period of time is defined here as "critical time," a definition that involves arbitrary bounds on the displacement as well as the magnitude and nature of the initial disturbance. Thus, unlike critical times defined in previous publications[10, 14], the critical time introduced here depends not only on the load and material properties of the column but also on the disturbance. By suitable nondimensionalisation, the effect of the magnitude of the disturbance on this critical time can be eliminated.

A dynamic analysis of viscoelastic simply supported perfect columns was carried out in [16] where some numerical results were obtained for a selected set of viscoelastic parameters. In this study, a closed-form solution in the form of a rapidly converging series is derived, which allows examination of this problem from a more general viewpoint for columns made of a three-element material.

2. DYNAMIC APPROACH

Consider an unloaded column at rest for time $t < 0$. The load P, which is constant in time, is applied at $t = 0$, while simultaneously the column is disturbed laterally, imparting to it a deflection, $w = w(x, t)$ (see Fig. 1). For $t > 0$, the behaviour of the column is defined by the following set of equations:

(i) Equation of motion

$$\frac{\partial^2 M}{\partial x^2} - P\frac{\partial^2 w}{\partial x^2} = m\frac{\partial^2 w}{\partial t^2}, \tag{1}$$

where

$$M(x, t) = \int_A \sigma(x, y, t) y \, dA, \qquad P = \int_A \sigma(x, y, t) \, dA$$

and where m denotes mass per unit length.

(ii) Constitutive law

$$\sigma(x, t) = E_0\left[\epsilon(x, t) + \int_0^t g(t - \tau)\epsilon(x, \tau) \, d\tau\right], \tag{2}$$

where the kernel, $g(t - \tau)$, defines the relaxation properties of the material.

(iii) Geometric relations (Bernoulli-Euler hypothesis)

$$\epsilon(x, y, t) = \epsilon_0(x, t) + y\kappa(x, t) \cong \epsilon_0(x, t) - y\frac{\partial^2 w(x, t)}{\partial x^2}. \tag{3}$$

Combining the above three relations, one obtains the equation of motion for the disturbed structure as

$$E_0 I\left[\frac{\partial^4 w}{\partial x^4} + g(t) * \frac{\partial^4 w}{\partial x^4}\right] + P\frac{\partial^2 w}{\partial x^2} + m\frac{\partial^2 w}{\partial t^2} = 0, \tag{4}$$

where (*) indicates convolution.

Assuming the solution to eqn (4) in the form

$$w(x, t) = \sum_{n=1}^{\infty} F_n(t) \sin\frac{n\pi x}{L}, \tag{5}$$

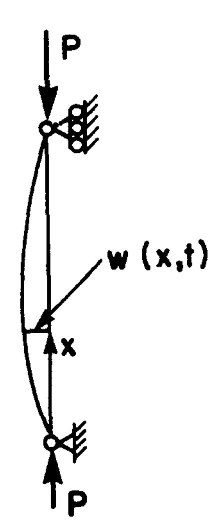

Fig. 1. The "disturbed" simply supported viscoelastic column.

one arrives at the following:

$$\frac{d^2F_n}{dt^2} + \omega_n^2\left(1 - \frac{P}{P_n^e}\right)F_n + \omega_n^2\, g(t) * F_n = 0 \tag{6}$$

subject to the initial conditions

$$F_n(0) = \mathring{F}_n \qquad \left.\frac{dF_n}{dt}\right|_{t=0} = \mathring{v}_n, \tag{7}$$

where

$$P_n^e = n^2\frac{\pi^2 E_0 I}{L^2} = n^2 P_E$$

$$n = 1, 2, 3, \ldots$$

$$\omega_n^2 = n^4\frac{\pi^4}{L^4}\frac{E_0 I}{m} = n^4\omega_0^2.$$

Applying the Laplace transform to eqn (6), one obtains

$$\bar{F}_n(s) = \frac{s\mathring{F}_n + \mathring{v}_n}{s^2 + \omega_n^2(1 - P/P_n^e) + \omega_n^2\, g(s)}, \tag{8}$$

where $\bar{F}_n(s)$ denotes the Laplace transform of the deflection parameter, which cannot be obtained without defining the creep behaviour of the material, i.e. defining the function $g(s)$.

3. SOLUTION FOR THE THREE-ELEMENT MATERIAL MODEL

The three-element model shown in Fig. 2 is the simplest spring-dashpot model that can simulate the behaviour of linear viscoelastic materials of the "solid" type with limited creep deformations when E_2 is nonzero, and of the "fluid" type with unlimited viscous deformations for $E_2 = 0$.

The kernel appearing in eqn (2) for this model as well as its Laplace transformation can be determined as

$$g(t) = -\frac{E_1}{\nu_2}e^{-\lambda t} = -(\lambda - \mu)e^{-\lambda t} \qquad g(s) = -\frac{\lambda - \mu}{s + \lambda}, \tag{9}$$

where, after [15], the coefficients λ and μ are given by

$$\lambda = \frac{E_1 + E_2}{\nu_2}, \qquad \mu = \frac{E_2}{\nu_2}. \tag{10}$$

Substituting (10) into (8), one obtains

$$\bar{F}_n(s) = \frac{(s\mathring{F}_n + \mathring{v}_n)(s + \lambda)}{s^3 + \lambda s^2 + \omega_n^2(1 - P/P_n^e)s - \omega_n^2\lambda(P/P_n^e - \mu/\lambda)}. \tag{11}$$

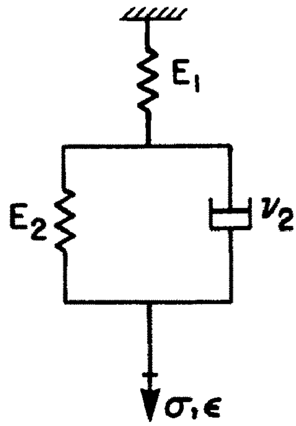

Fig. 2. The three-element viscoelastic model.

The denominator of eqn (11) has three roots, s_1, s_2 and s_3. For $P < P_n^e$, s_1 is real while s_2 and s_3 are complex, and the denominator can be expressed in the form

$$s^3 + \lambda s^2 + \omega_n^2\left(1 - \frac{P}{P_n^e}\right)s - \omega_n^2\lambda\left(\frac{P}{P_n^e} - \frac{\mu}{\lambda}\right) = (s - S_1^{(n)}) \times [s - (\alpha_n + i\beta_n)][s - (\alpha_n - i\beta_n)]. \quad (12)$$

Expressions for $S_1^{(n)}$, α_n and β_n can be obtained in general, but their form is complicated. Expanding $S_1^{(n)}$, α_n and β_n into a power series of the small dimensionless parameter $\delta_n = \lambda/\omega_n$, one obtains

$$S_1^{(n)} = \lambda\frac{P/P_n^e - \mu/\lambda}{1 - P/P_n^e}\left[1 - \frac{(P/P_n^e - \mu/\lambda)(1 - \mu/\lambda)}{(1 - P/P_n^e)^3}\delta_n^2 + o(\delta_n^4)\right] \quad (13a)$$

$$\alpha_n = -\frac{\lambda}{2}\frac{1 - \mu/\lambda}{1 - P/P_n^e}\left[1 + \frac{(P/P_n^e - \mu/\lambda)^2}{(1 - P/P_n^e)^3}\delta_n^2 + o(\delta_n^4)\right] \quad (13b)$$

$$\beta_n = \sqrt{1 - \frac{P}{P_n^e}}\,\omega_n\left\{1 + \frac{1}{2}\frac{1 - \mu/\lambda}{(1 - P/P_n^e)^3}\left[\frac{P}{P_n^e} - \frac{1}{4}\left(3\frac{\mu}{\lambda} + 1\right)\right]\delta_n^2 + o(\delta_n^4)\right\}, \quad (13c)$$

where the parameter δ_n can be expressed in the form

$$\delta_n = \frac{1}{2\pi}\frac{T_n}{t_r}$$

and where T_n is the period of elastic "free vibration" for the nth mode and

$$t_r = \frac{\nu_2}{E_1 + E_2}$$

denotes the relaxation period. For real materials, δ_n is very small: for example, for concrete, $\delta_n \cong 1.2 \times 10^{-8}/n^2$[16]. This means that the values for $S_1^{(n)}$, α_n and β_n can be obtained with sufficient accuracy from the first term in eqns (13), neglecting all terms involving powers of δ_n.

Equation (11) can be rewritten as

$$F_n(s) = \frac{A_n}{s - S_1^{(n)}} + \frac{B_n s + C_n}{(s - \alpha_n)^2 + \beta_n^2}, \quad (14)$$

where

$$
\begin{aligned}
A_n &= (\mathring{v}_n + S_1{}^{(n)}\mathring{F}_n)\frac{-2\alpha_n}{\alpha_n^2 + \beta_n^2 + S_1{}^{(n)}(-2\alpha_n + S_1{}^{(n)})} \\
&\cong \delta_n\left(\frac{\mathring{v}_n + S_1{}^{(n)}\mathring{F}_n}{\omega_n}\right)\frac{1 - \mu/\lambda}{(1 - P/P_n^e)^2} \\
B_n &= \mathring{F}_n - A_n \\
C_n &= A_n\frac{\alpha_n^2 + \beta_n^2}{S_1{}^{(n)}} - \lambda\frac{\mathring{v}_n}{S_1{}^{(n)}}.
\end{aligned}
\tag{15}
$$

Applying the inverse Laplace transformation to eqn (14), one obtains

$$
F_n(t) = A_n\,e^{\bar{s}_n\bar{t}} + e^{\bar{\alpha}_n\bar{t}}\,[(\mathring{F}_n - A_n)\cos\bar{\beta}_n\bar{t} + D_n\sin\bar{\beta}_n\bar{t}],
\tag{16}
$$

where

$$
D_n = \frac{\mathring{v}/n - \mathring{F}_n\,\bar{\alpha}_n + A_n(\bar{\alpha}_n + \bar{s}_n)}{\bar{\beta}_n}
$$

and where the following dimensionless quantities have been used:

$$
\begin{aligned}
&\bar{t} = \lambda t, \qquad \bar{p} = \frac{P}{P_n^e}, \qquad \bar{P}_v = \frac{\mu}{\lambda} = \frac{E_2}{E_1 + E_2} \\
&\bar{s}_n = \frac{S_1{}^{(n)}}{\lambda} = \frac{\bar{p}_n - \bar{p}_v}{1 - \bar{p}_n}(1 - \ldots) \\
&\bar{\alpha}_n = \frac{\alpha_n}{\lambda} = -\frac{1}{2}\frac{(1 - \bar{p}_v)}{1 - \bar{p}_n}(1 + \ldots) \\
&\bar{\beta}_n = \frac{\beta_n}{\lambda} = \frac{\sqrt{1 - \bar{p}_n}}{\delta_n}(1 + \ldots).
\end{aligned}
\tag{17}
$$

Note that $\bar{s}_n \ll \bar{\beta}_n$, $\bar{\alpha}_n \ll \bar{\beta}_n$, $A_n \ll \mathring{F}_n$ and $A_n \ll D_n$.

For $P > P_n^e$, all three roots of the denominator of (11) are real and the solution will consist of three exponential functions in the form

$$
F_n(t) = A_n\,e^{s_1 t} + K_2\,e^{s_2 t} + K_3\,e^{s_3 t}.
\tag{18}
$$

4. DISCUSSION OF STABILITY

The response of a viscoelastic column to disturbances represented by the functions (16) or (18) depends on the values of the parameters $\bar{s}_n$ and $\bar{\alpha}_n$.

For $\bar{p} > 1$, i.e. $P > P_n^e$, $F_n(t)$ is a positive exponential function of time and the structure is obviously unstable. Let us refer to this type of instability as "dynamic instability."

For $\bar{p} < \bar{p}_v$, i.e. $P < [P_E/(1 + E_1/E_2)$, $\bar{s}_n < 0$ and $\bar{\alpha}_n < 0$ and therefore $F_n(t) \to 0$ as $t \to \infty$, irrespective of the type of disturbance. Thus, the column is stable and the value of the load

$$
P_v = \frac{P_E}{1 + E_1/E_2}
$$

will be called the "viscoelastic critical force," or the safe load limit[14].

If $\bar{p}_v < \bar{p} < 1$, i.e. $P_v < P < P_E$, $\bar{s}_n > 0$ and $\bar{\alpha}_n < 0$ and $F_n(t) \to \infty$ for $t \to \infty$, as in the dynamic instability case. However, since $|\bar{s}_n| < |\bar{\alpha}_n|$ and $|A_n| \ll |\mathring{F}_n|$ and $|A_n| \ll |D_n|$, the second term in eqn (16), representing the damped free vibration of the disturbed viscoelastic column, will decay much more rapidly than the first term in that equation will grow with time. In fact, the increase in the first term, owing to the smallness of A_n, will be very slow, at least initially. In time, this term, which contains the small but significant "memorised" portion of the initial disturbance, will grow and will ultimately be responsible for the failure of the structure. Thus, although the column is unstable in a dynamic sense, when $P_v < P < P_E$, it may fulfill a useful function during a limited time period after loading, a period defined here as the critical time. This type of instability will be referred to as "viscoelastic instability."

The phase plane diagram shown in Fig. 3 indicates the three types of behaviour discussed above. In Fig. 3a, M_s, M_D and M_m denote static, dynamic and mixed initial

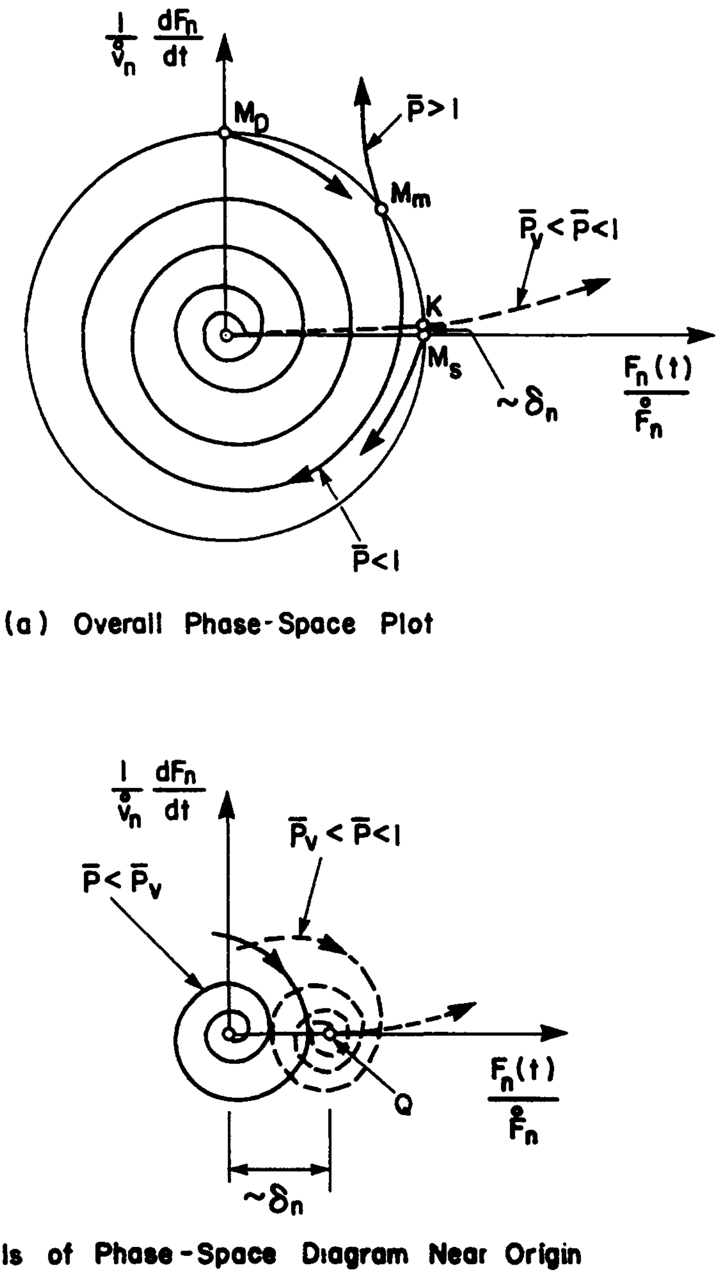

Fig. 3. Phase-space diagram for viscoelastic perfect column. (a) Overall phase-space plot. (b) Details of phase-space diagram near origin.

disturbances, respectively. Clearly, for $\bar{p} > 1$, the response to any disturbance diverges dynamically. For $\bar{p} < 1$, the response decays to a point close to the origin, the details of which are indicated on Fig. 3b. This enlargement of the area in the vicinity of the origin indicates that for $\bar{p} < \bar{p}_{\nu}$, the response, indeed, decays to zero, while for $\bar{p}_{\nu} < \bar{p} < 1$, the response decreases in time to some small deflection of the order of magnitude δ_n, which represents the memorised portion of the initial disturbance and which slowly, but steadily, grows in time and ultimately is responsible for the instability of the structure. As is clear from eqn (15), A_n and therefore the time taken for the deflection to grow beyond the magnitude of the initial disturbance are functions of the type and magnitude of the disturbance.

Note that the critical loads P_E and P_{ν} have been determined only in terms of the elastic properties of the material; the Euler buckling force corresponds to the initial instantaneous elasticity ($t = 0$) of the model, when the viscous mechanism is still inactive (see Fig. 4a). The force P_{ν}, however, is related to the final elasticity ($t \to \infty$),

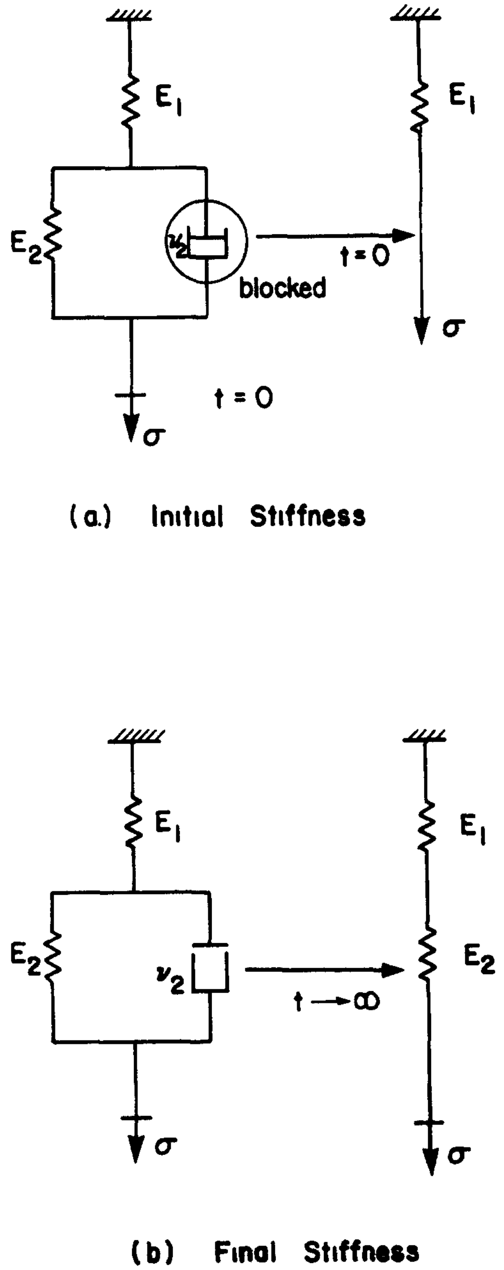

Fig. 4. Initial (a) and final (b) stiffness of three-element model.

when the dashpot does not provide any more resistance (see Fig. 4b). The same conclusion can also be drawn for models of the fluid type, assuming that $E_2 \to 0$. However, the stiffness of such models tends to zero as $t \to \infty$, and therefore $P_v = 0$ for such materials.

5. ANALYSIS OF VISCOELASTICALLY UNSTABLE COLUMNS

Recall that we define viscoelastic instability of such structures to be the instability for the load levels $P_v < P < P_E$. Under such loads, the structure may fulfill a useful role, as noted earlier. Let us examine this particular instability domain in detail.

5.1 *Static disturbances*

First, investigate the behaviour of such a column when subjected to a purely static disturbance, i.e. $\mathring{v}_n = 0$ and $\mathring{F}_n \neq 0$ (see point M_s in Fig. 3a). Therefore, from eqns (13)–(17), one obtains

$$F_n(t) = \mathring{F}_n \{\delta_n^2 a_n e^{\bar{s}_n \bar{t}} + e^{\bar{\alpha}_n \bar{t}} [(1 - \delta_n^2 a_n) \cos \bar{\beta}_n \bar{t} + \delta_n d_n \sin \bar{\beta}_n \bar{t}]\} \tag{19}$$

$$A_n^s = -\mathring{F}_n \frac{2\alpha_n s_n}{\beta_n^2 + \alpha_n^2 + s_n(-2\alpha_n + s_n)} = \mathring{F}_n \delta_n^2 a_n(\bar{p})$$

$$B_n^s = \mathring{F}_n[1 + o(\delta_n^2)] \tag{20}$$

$$D_n^s = \mathring{F}_n \delta_n d_n(\bar{p}),$$

where

$$a_n(\bar{p}) = \frac{(1 - \bar{p}_v)(\bar{p} - \bar{p}_v)}{(1 - \bar{p})^2} [1 + o(\delta_n^2)]$$

$$d_n(\bar{p}) = \frac{1}{2} \frac{(1 - \bar{p}_v)}{(1 - \bar{p})^{3/2}} [1 + o(\delta_n^2)].$$

Taking values of parameters as used in [16], $\lambda = 2\mu = 5.78 \times 10^{-7}$/sec and $\omega_1 = 48$ rad/sec, we have

$$\delta_1 = 1.206 \times 10^{-8}, \qquad \bar{p}_v = 0.5.$$

Using these data and assumed values for $\bar{p}$, we obtain

	$\bar{p} = 0.6$	$\bar{p} = 0.8$
a_1	0.313	3.750
d_1	0.988	2.795
$\bar{\alpha}_1$	−0.625	−1.251
$\bar{s}_1$	0.250	1.500
$\bar{\beta}_1$	5.246×10^7	3.709×10^7

The deflection function given by eqn (19) is sketched in Fig. 5.

Clearly, the vibrational part of the motion is damped out after $\bar{t} \gtrsim 5$, while the memorised disturbances become significant only after $t \gtrsim 20$ for $\bar{p} = 0.8$ and after $t \gtrsim 150$ for $\bar{p} = 0.6$. There is obviously quite a long intermediate period, when the response amplitude of the structure is extremely small, $F_1/F_0 \cong 10^{-7}$, a consequence of the very small value of the memorised portion of the disturbance that is proportional to δ_n^2 for this case [see eqn (20)].

5.2 *Dynamic disturbances*

Next, analyse the response of the structure when subjected to a dynamic disturbance (point M_D in Fig. 3a), i.e. $\mathring{F}_n = 0$ and $\mathring{v}_n \neq 0$. From eqns (13)–(17) one obtains,

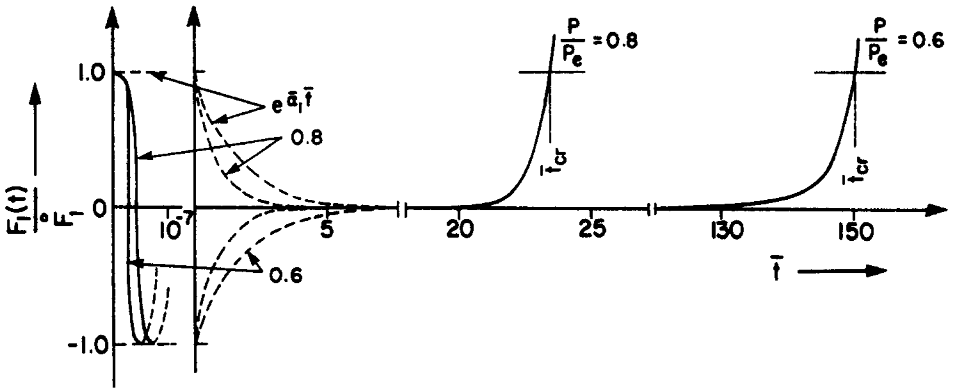

Fig. 5. Deflection history for column subjected to "static" disturbances.

again,

$$F_n(t) = \frac{\mathring{v}_n}{\beta_n}\left[\delta_n g_n\, e^{\bar{s}_n \bar{t}} + e^{\bar{\alpha}_n \bar{t}}\left(\sin \bar{\beta}_n \bar{t} - \delta_n g_n \cos \bar{\beta}_n \bar{t}\right)\right] \tag{21}$$

$$A_n = -\mathring{v}_n \frac{2\alpha_n}{\alpha_n^2 + \beta_n^2 + s_n(-2\alpha_n + s_n)} = \frac{\mathring{v}_n}{\beta_n}\delta_n\, g_n(\bar{p}) \tag{22a}$$

$$B_n = -A_n \tag{22b}$$

$$D_n = \frac{\mathring{v}_n}{\beta_n}\left[1 + o(\delta_n^2)\right], \tag{22c}$$

where

$$g_n(\bar{p}) = \frac{1 - \bar{p}_v}{(1 - \bar{p})^{3/2}}\left[1 + o(\delta_n^2)\right].$$

Note that now the memorised part of the disturbance is proportional to δ_n, while for the static disturbance it was proportional to δ_n^2.

To evaluate the effects of these two kinds of disturbances, let us assume that the energy of the disturbances is the same for both cases; i.e. for the dynamic disturbance, the kinetic energy is given as

$$K = \int_0^L \frac{m}{2}\left[\frac{\partial w(x, 0)}{\partial x}\right]^2 \mathrm{d}x = \frac{m}{2}\sum_n \mathring{v}_n^2 \frac{L}{2},$$

while for the static disturbance, the change in potential energy in going from the undisturbed to the disturbed state is written as

$$E_s = \int_0^L \frac{E_0 I}{2}\left[\frac{\partial^2 w(x, 0)}{\partial x^2}\right]^2 \mathrm{d}x - \frac{P}{2}\int_0^L \left[\frac{\partial w(x, 0)}{\partial x}\right]^2 \mathrm{d}x$$
$$= \frac{E_0 I}{2}\frac{L}{2}\sum_n \mathring{F}_n^2 \left(\frac{n\pi}{L}\right)^4\left(1 - \frac{P}{P_n^e}\right).$$

Therefore,

$$m\,\mathring{v}_n^2 = E_0 I\,\mathring{F}_n^2 \left(\frac{n\pi}{L}\right)^4\left(1 - \frac{P}{P_n^e}\right) \tag{23}$$

or

$$\overset{\circ}{v}_n = \overset{\circ}{F}_n\, \omega_n \sqrt{1 - \frac{P}{P_n^c}} = \overset{\circ}{F}_n\, \beta_n,$$

which, when substituted into eqn (21), gives

$$F_n(t) = \overset{\circ}{F}_n\, [\delta_n g_n\, e^{\bar{s}_n \bar{t}} + e^{\bar{\alpha}_n \bar{t}}\, (\sin \bar{\beta}_n \bar{t} - \delta_n g_n \cos \bar{\beta}_n \bar{t})]. \qquad (24)$$

For data analogous to those used above, a plot of eqn (24) is shown in Fig. 6.

The overall character of the diagrams shown in Figs. 5 and 6 is similar. Clearly, the "intermediate" period for the dynamic case is considerably shorter than that for the static disturbance case, owing to the fact that the memorised portion of the disturbances is bigger by a factor of δ_n^{-1}, i.e. by a factor of $\approx 8.3 \times 10^7$.

On the basis of Figs. 5 and 6, the response of such a structure can be divided into three main parts: (i) the damped vibration; (ii) the intermediate period, with the response amplitude ≈ 0; and (iii) the period of rapid growth in amplitude. This type of behaviour, as noted already, is characteristic only for the structure with the load range $P_v < P < P_e$.

5.3 *The critical time*

Having determined the main features of the behaviour of such columns, one can define a critical time. For example, one might define such a period as the time required for the response amplitude to become equal to the magnitude of the initial disturbance (point K in Fig. 3). Such a definition for the critical time is independent of the value of the initial disturbance, whether it be static or dynamic (see Figs. 5 and 6). In general, such a critical time may be found from the relation

$$\sum_n A_n\, e^{\bar{s}_n \bar{t}_{cr}} \cong \sum_n \sqrt{(\overset{\circ}{F}_n - A_n)^2 + D_n^2}. \qquad (25)$$

Because values of successive coefficients in this series are smaller by the factor n^{-2}, it is sufficient for practical purposes to take into account only the first term. The higher harmonics will theoretically affect the critical time only for $\bar{s}_n > 0$, which corresponds to the condition

$$n^2 \frac{\mu}{\lambda} < \frac{P}{P_E} < 1,$$

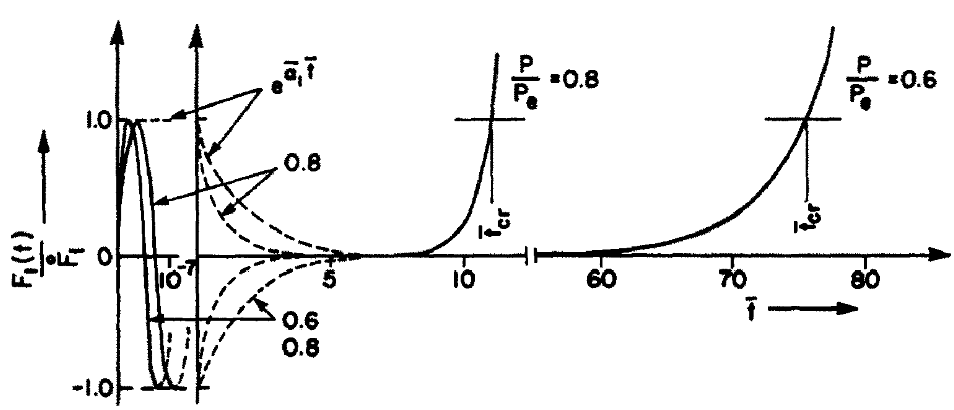

Fig. 6. Deflection history for column subjected to "dynamic" disturbances.

which in turn means that $n^2 < \mu/\lambda$ or $n^2 < E_1/E_2 + 1$. Thus, if $E_1 < 3E_2$, only the first harmonic has to be taken into account, while for $E_1 = 19E_2$, for example, all four harmonics should be included.

Using only the first term in eqn (25), the critical time can be determined explicitly as follows:

(i) For static disturbances:

$$\bar{t}_{cr}{}^{s} \cong -\frac{\ln(a_1\delta_1^2)}{\bar{s}_1} \tag{26}$$

(ii) For dynamic disturbances:

$$\bar{t}_{cr}{}^{d} \cong -\frac{\ln(g_1\delta_1)}{\bar{s}_1}, \tag{27}$$

expressions that for the data used above are plotted in Fig. 7. From Figs. 5 and 6, for $P/P_E = 0.8$,

$$\bar{t}_{cr}{}^{s} = 22.35$$

and

$$\bar{t}_{cr}{}^{d} = 11.0.$$

Comparing eqns (26) and (27), and since $|\ln a_1/\ln \delta_1| \ll 1$ and $|\ln g_1/\ln \delta_1| \geq 1$, one obtains

$$\frac{\bar{t}_{cr}{}^{s}}{\bar{t}_{cr}{}^{d}} = \frac{2 + \ln a_1/\ln \delta_1}{1 + \ln g_1/\ln \delta_1} \approx 2. \tag{28}$$

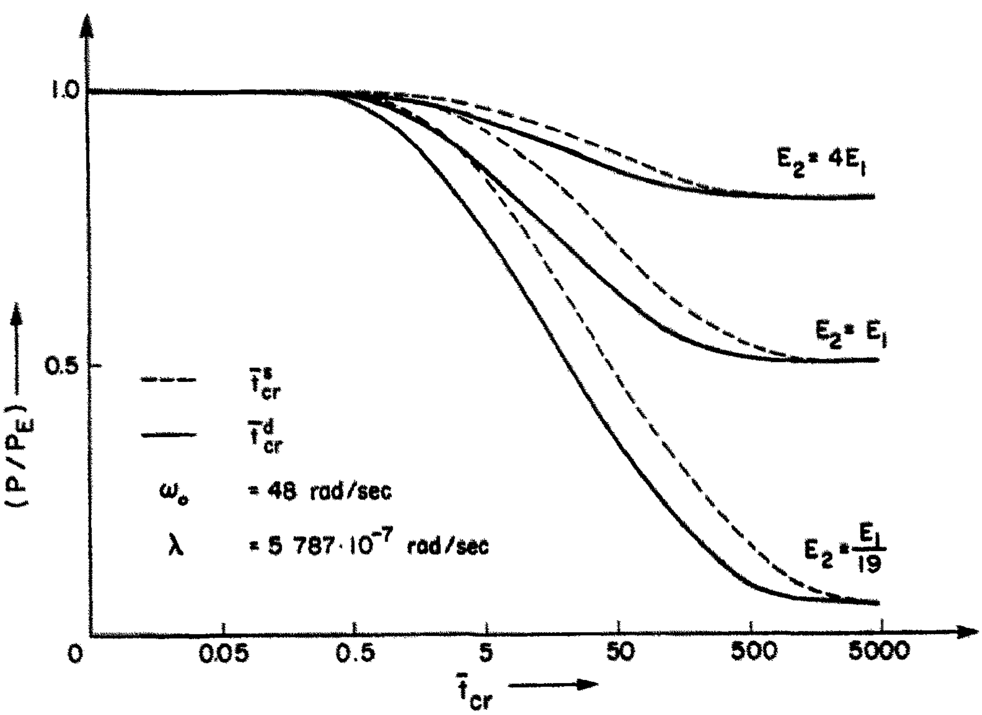

Fig. 7. Effect of axial load on "critical time" for various model stiffnesses.

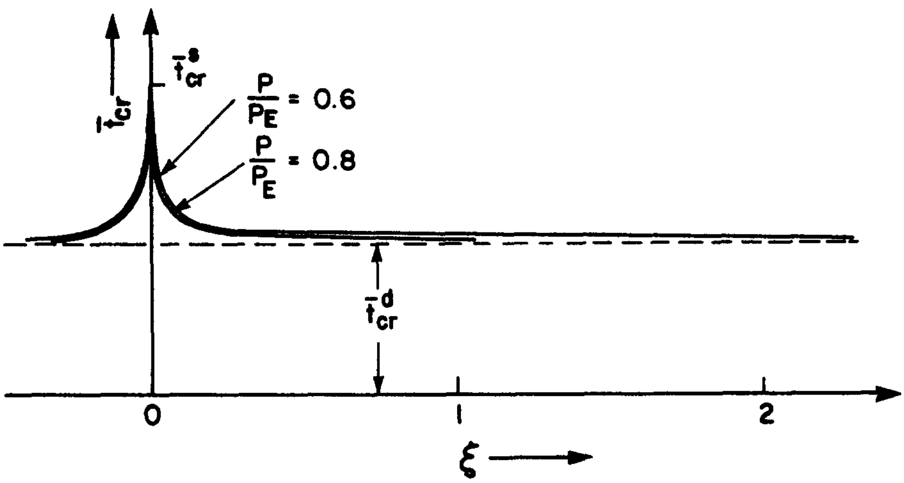

Fig. 8. "Critical time" for mixed disturbances.

For the case of "mixed" disturbances, assume a relation of the form

$$\mathring{v}_n = \xi\beta_n\mathring{F}_n, \qquad 0 \leq |\xi| \leq \infty, \tag{29}$$

where ξ represents the "ratio" of static and dynamic disturbances. Thus, $\xi = 0$ represents the static case, and $\xi \to \infty$ denotes the dynamic case. Substituting this relation into eqn (23) and retaining only the first harmonic, one obtains

$$\bar{t}_{cr} = \frac{\ln\{[\bar{\beta}_1^2/2\bar{\alpha}_1(\xi\bar{\beta}_1 + \bar{s}_1)]^2 + [\bar{\beta}_1/2(\xi\bar{\beta}_1 + \bar{s}_1) - (\bar{\alpha}_n + \bar{s}_1)/\bar{\beta}_1 + \xi\bar{\beta}_1^2/(\xi\bar{\beta}_1 + \bar{s}_1)2\bar{\alpha}_1]\}}{2\bar{s}_1}, \tag{30}$$

a relation that has been numerically proven to be a monotonic function of $|\xi|$ (Fig. 8). Thus, for the mixed disturbances, the critical time can be estimated from

$$\bar{t}_{cr}{}^{d} < \bar{t}_{cr} < \bar{t}_{cr}{}^{s}, \tag{31}$$

where the lower and upper bounds to this estimate are independent of the value of the disturbance.

6. CONCLUSIONS

A dynamic stability analysis of perfect columns made of a linear three-element model material and subjected to constant concentric axial loads has been carried out. The analysis shows that for a column made of such a material, there exists a so-called viscoelastic critical force, also referred to as "safe load limit," below which the equilibrium of the column is stable at all times. Interestingly, this safe load limit, like the Euler load, is a function of only the elastic properties of the material, but while the Euler buckling load is a function of the "initial stiffness" of the model, the viscoelastic critical force is a function of the "final stiffness" of the material. Since for a fluid-type material the final stiffness tends to zero, the viscoelastic critical force for columns made of such a material also is zero.

For loads in excess of the Euler buckling load, the column naturally is dynamically unstable. More interesting is the response for loads ranging between the safe load limit and the Euler load. It is shown that although for this load range the stability of the equilibrium of the column is also unstable, there is a period, defined herein as the "critical time," during which the lateral deflection of the column remains within a prescribed bound, and thus the column may perform a useful function for a limited time interval. The definition of such a critical time does not depend only on the material properties and geometry of the column, but also on an arbitrary bound on the magnitude of lateral deflections as well as on the type and magnitude of the initial disturbance. By suitable nondimensionalisation, the effect of the magnitude of initial disturbances on this definition can be eliminated. Thus, this definition for critical time is subjective and may have only limited use in practical applications.

The results also indicate that this critical time is very much a function of the type of disturbance to which the initially perfect column is subjected. Dynamic disturbances will be twice as harmful, from this point of view, as corresponding static disturbances. Thus, the critical time defined here is twice as long for instability precipitated by static disturbances as by equivalent dynamic disturbances. It is also shown that the critical time associated with a mixed disturbance, composed of both static and dynamic disturbances, is bounded by the critical times of the static and dynamic responses, respectively.

On the basis of this investigation, one concludes that such a viscoelastic column is stable when subjected to an axial load $P < P_v$. For loads ranging between $P_v < P < P_E$, the response of the column to lateral disturbances consists of three main parts: (i) an initial damped free vibration, during which the response amplitude decreases rapidly and approaches zero; (ii) a relatively long intermediate period, following the initial vibrational portion, during which the response amplitude is approximately equal to zero; and (iii) a period of rapid growth in the response amplitude during which the memorised portion of original disturbance starts to grow rapidly, leading ultimately to unbounded displacements and failure of the structure.

It is the memorised portion of the disturbance signal that is retained by the material and that ultimately causes the collapse of the column. Thus, here we have yet another example of the large class of materials "with memory," for which the past history of the structure is significant in determining its present and future response.

In summary, one should note that all results and conclusions obtained here refer to columns made of a linear three-element model material. However, despite its simplicity, this model is being used and can simulate the behaviour of some materials under certain specific loading and/or environmental conditions, as, for example, concrete or ice under low stress levels. Also, the approach used here is, of course, applicable to columns made of materials with more complex constitutive laws; the tractability of such problems and the degree of success in their solution will depend on a great many factors, including the experience and ingenuity of the investigator. The results obtained here do provide some insight into the stability behaviour of such structures and should be helpful in further studies.

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